1. Relying upon the well-known theorem from quantum mechanics on simultaneous diagonalizability of commuting Hermitian operators, show that the (non-Hermitian) translation operator 
\[ \hat{T}_{d} = e^{d \cdot \nabla} \]
has a complete orthonormalized set of eigenfunctions in \( L^{2}(\mathbb{R}^{3}) \) that are also eigenfunctions of the Hamiltonian 
\[ \hat{H} = \frac{\hat{p}^{2}}{2m} + V(x), \]
with \( V(x + d) = V(x) \).

2. Trans-polyacetylene \((\text{CH})_{n}\) is a linear polymer, which is known to develop a conjugated (delocalized) orbital from one \( p_z \) orbital per each of the carbon atoms. It is also known to dimerize, i.e., exist in the form with alternating ‘single’ (weaker) and ‘double’ (stronger) bonds \((-\text{CH} = \text{CH}-)_{n}\). From the tight-binding standpoint, such a system is described by the Su–Schrieffer–Heeger (SSH) model with the Hamiltonian
\[ \hat{H} = \sum_{n=\infty}^{\infty} \left\{ u \hat{c}_{2n}^{\dagger} \hat{c}_{2n+1} + v \hat{c}_{2n}^{\dagger} \hat{c}_{2n-1} \right\}, \]
featuring the ‘single-bond’ and ‘double-bond’ hopping integrals \( u, v \). Find the energy dispersion relation \( \varepsilon(k) \) for this TB model \((\text{Note: mind the unit cell first!})\) and the effective mass for electrons \((\varepsilon > 0)\).

3. Find the Brillouin zone for a 2D honeycomb lattice \((\text{lattice parameter } d)\) and, by translating certain pieces of it by reciprocal-lattice vectors, transform it into a hexagon.

4. Find the ground-state energy of a helium atom using the Hartree–Fock–Roothaan equations based on (a) \( 1s \) orbitals, (b) orbitals with \( n = 1, \ldots, n_{\text{max}} \) for some reasonable \( n_{\text{max}} \); (c) using the post-Hartree–Fock configuration interaction scheme. Compare the energies and the RMS charge radii \( \sqrt{\frac{1}{2} \int \rho(r) d^{3}x} \) thus obtained \((\rho(x) \text{ is the total charge density of two electrons, in units of } e)\).

5. Find the density profile \( n(r) \) for an atom with the nucleus charge \(-Ze\) using the Thomas–Fermi equation. Note: resort to spherically-symmetric solutions; find the form of the \( r \to \infty \) and the \( r \to 0 \) asymptotics; estimate the unknown parameters in the asymptotic expressions numerically.

6. Find the leading coefficient \( g_{0} \) in the exchange-correlation energy for the dilute homogeneous electron gas of density \( n \),
\[ \epsilon_{\text{xc}} = g_{0} n^{1/3} + \mathcal{O}(n^{1/2}), \quad n \to 0. \]

7. Show that for a lattice with the primitive cell containing only one lattice site, the electron Green’s function in the tight-binding model is
\[ G_{\sigma\sigma'}(x, y, t) = -i \langle \text{FS} | \hat{T}_{e_{\sigma}}(x, t) \hat{c}_{\sigma'}^{\dagger}(y, 0) | \text{FS} \rangle = \delta_{\sigma, \sigma'} \sum_{k \in \text{BZ}} e^{ik(x-y)} \frac{\epsilon(k)}{2\pi} \frac{2\pi}{\Omega} \left\{ \frac{\theta(\epsilon_k - \epsilon_F) - \theta(\epsilon_F - \epsilon_k)}{\omega - \epsilon_k + i0} + \frac{\theta(\epsilon_F - \epsilon_k)}{\omega - \epsilon_k - i0} \right\}, \]
where \( \Omega \) is the total number of the lattice sites \((= \text{the number of points in the Brillouin zone}), \epsilon_k \text{ is the spectrum of 1-particle energies, and } \epsilon_F \text{ is the Fermi energy.} \text{ Note: prove and then use the identity}
\[ \theta(t)e^{-i\Omega t} = \int_{-\infty}^{+\infty} \frac{i d\omega}{2\pi} \frac{e^{-i\omega t}}{\omega - \Omega + i0}. \]
8. Let us look at a hydrogen $H_2$ molecule from a Hubbard-model standpoint

$$\hat{H} = -t(d) \sum_{\sigma=\uparrow,\downarrow} \left\{ \hat{c}^\dagger_{\sigma i} \hat{c}_{\sigma i} + \hat{c}^\dagger_{\sigma i} \hat{c}_{\sigma i} \right\} + U \sum_{i=1,2} \hat{n}_{\uparrow i} \hat{n}_{\downarrow i}, \quad N = 2.$$  

Here, we have only two sites $i = 1, 2$ and four creation operators $\hat{c}^\dagger_{\sigma i}$; the transfer integral strongly depends on the orbital overlap, so it is assumed to depend on the distance $d$ between the two nuclei. Using the fact four operators $\hat{S}^2 = (\hat{S}_1 + \hat{S}_2)^2, \hat{S}_z = \hat{S}_{1z} + \hat{S}_{2z}, \hat{P}, \hat{N} = \sum_{i,\sigma} \hat{c}^\dagger_{\sigma i} \hat{c}_{\sigma i}$ commute with each other and with $\hat{H}$, find the energies and wave functions for triplet $S = 1$ and singlet $S = 0$ states of the molecule. The spatial parity operator simply exchanges the two sites, $\hat{P} \hat{c}_{\sigma i} \hat{P} := \hat{c}_{3-i,\sigma}$. Compare your results with the TB $(U = 0)$ case.

9. Find the divergent part of the density of states for a tight-binding model on a 2D square lattice with all NN hopping integrals equal to $t$

$$\rho(\epsilon) = \frac{N}{2\pi t} \log \frac{t}{|\epsilon|} + \mathcal{O}(1) \quad \text{(lattice units, } d = 1).$$

This Van Hove singularity takes place exactly at the Fermi surface in the case of half-filling.

10. Apply a Bogolyubov transformation to the mean-field BCS Hamiltonian

$$\hat{H} = -\mu \hat{N} = \sum_{k\sigma} (\epsilon_k - \mu) \hat{c}^\dagger_{k\sigma} \hat{c}_{k\sigma} - \sum_k \left\{ \Delta^* \hat{c}^\dagger_{-k\uparrow} \hat{c}_{k\downarrow} \right\} + \frac{V}{a^3 U_0} |\Delta|^2,$$

$$\hat{c}_{k\uparrow} = u_k \hat{c}_{k\uparrow} + v_k \hat{c}^\dagger_{-k\downarrow},$$

$$\hat{c}^\dagger_{-k\downarrow} = -v^*_k \hat{c}^\dagger_{-k\downarrow} + u_k \hat{c}^\dagger_{k\uparrow},$$

where $|u_k|^2 + |v_k|^2 = 1$. By finding the appropriate $u_k, v_k$ coefficients, eliminate the particle number violating terms and find the quasiparticle energies as a function of $\Delta$.

11. Consider strained graphene with the Hamiltonian

$$\hat{H} = -\sum_{x \in A} \sum_{j=1,2,3} (t + \Delta t_j) \hat{b}^\dagger_{x}(x + \delta_j) \hat{a}_\sigma(x) + \text{h.c.}, \quad (1)$$

where $\delta_{1,2,3}$ connect a site $x$ of sublattice $A$ with its nearest neighbours of sublattice $B$ and $|\Delta t_j| \ll t$ (in general, $\Delta t_j \in \mathbb{C}$). In the leading order in $|\Delta t_j|/t$, find the shifted Fermi points (and justify that the Fermi surface is still degenerate, consisting of only two points). Introducing the wavefunction $\Psi_\sigma(x)$ containing the low-energy degrees of freedom, find the effective wave equation for it.

12. Consider a graphene sheet with a small Kekulé distortion of the lattice, which corresponds to

$$\Delta t_j \sim \Delta t_j(x) = \kappa \left\{ e^{i(Q x + k_+ \delta_j)} + e^{-i(Q x + k_- \delta_j)} \right\} \to 0, \quad Q \equiv k_+ - k_-,$$

in the above Hamiltonian $[1]$. Find the modification of the effective Dirac equation resulting from such a perturbation (N.B.: it will mix the two Fermi points $k_{\pm}$)

13. Find the one-particle eigenstates and their energies for the $\pi$-band electrons in graphene in the external magnetic field $B = B e_2$ orthogonal to the graphene plane (neglecting the Zeeman interaction $\sigma \mu B$). Use the potential $A(x) = e B x e_y$.

14. In the $\xi = 1$ gauge, integrate out the photon field $A_\mu$ in the partition function $Z[\xi_\sigma, \bar{\xi}_\sigma]$ for the $\pi$ electrons in graphene ($\xi_\sigma(x), \bar{\xi}_\sigma(x)$ are the sources conjugate to the spinor fields $\Psi_\sigma(x), \bar{\Psi}_\sigma(x)$, respectively). As a result, find the effective action for the interacting graphene and resort to its nonrelativistic limit

$$S_{\text{eff}}[\Psi_\sigma, \bar{\Psi}_\sigma] = \int d^3 x \bar{\Psi}_\sigma(x) i(\gamma^0 \partial_0 + \hbar \gamma \cdot \nabla) \Psi_\sigma(x) - \frac{e^2}{8\pi} \int \frac{dtd^2}{} \bar{\Psi}_\sigma(x) \gamma^0 \Psi_\sigma(x) \bar{\Psi}_\xi(y) \gamma^0 \Psi_\xi(y) \frac{x - y}{|x - y|}.$$